View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

NORTH-HOLLAND

Circuit Separation for Symmetric Matroids

Alain Duchamp
Département de Mathématiques et Informatique
Université du Maine
72017 Le Mans Cedex, France

Submitted by Richard A. Brualdi

ABSTRACT

Symmetric matroids and their associated structure (delta matroids) are a generalization of finite matroids introduced by A. Bouchet to extend several nice properties related to matroids (greedy algorithm, duality, representation). For a given symmetric matroid, the family $\mathscr{C}$ of its circuits is said to be *weakly separate* if the relation "$x, y \in C$ for some circuit C" on W is transitive, while $\mathscr{C}$ is said to be *separate* if there exists a transversal V of W such that $C \subseteq V$ or $C \cap V = \varnothing$ for any circuit C. These two classes of symmetric matroids are characterized by equivalent properties and excluded minors. Then we give the corresponding interpretation in delta matroids with the characterization of two types of delta matroids coming from matroids, and two results on the symmetric matroids considered as an intersection.

1. PRELIMINARIES AND DEFINITIONS

Symmetric matroids, delta matroids, g-matroids, metroids, and pseudomatroids are interesting and related generalizations of matroids, which were introduced at about the same time, in the mid eighties (see [1] to [5], and [11, 12]). The principal motivation of these generalizations has been to extend the greedy algorithm, the rank function of matroids, and Edmond's intersection theorem, or to analyze properties of Euler tours of 4-regular graphs and combinatorial relations defined by nonsingular principal minors of quasisymmetric matrices. With another starting point, I. M. Gelfand and V. V. Serganova [9] invented the concept of (W, P) matroids, which unifies the

LINEAR ALGEBRA AND ITS APPLICATIONS 231:87–103 (1995)

© Elsevier Science Inc., 1995
655 Avenue of the Americas, New York, NY 10010

0024-3795/95/$9.50
SSDI 0024-3795(94)00011-2

structures of matroids, delta matroids, and a large class of greedoids (see [10]).

We recall briefly the basic notions of the theory of delta matroids and symmetric matroids [1], assuming that the reader is familiar with the basic facts of matroid theory [14].

For a matroid $M = (V, \mathbf{B})$, where $\mathbf{B}$ is the base set of M, we have the *exchange axiom*

(EA) For $B_1, B_2 \in \mathbf{B}$ and $x \in B_1 \setminus B_2$, there exists some $y \in B_2 \setminus B_1$ such that $B_1 - x + y = B_1 \vartriangle \{x, y\} \in B$,

where $\vartriangle$ denotes the symmetric difference operator: $X \vartriangle Y = (X \cup Y) \setminus (X \cap Y)$.

Delta Matroids

A delta matroid on the finite set V is a set system $\delta = (V, \mathscr{F})$ such that the *base set* $\mathscr{F}$, $\mathscr{F} \neq \emptyset$, satisfies the *symmetric exchange axiom*,

(SEA) For $F_1, F_2 \in \mathscr{F}$ and $x \in F_1 \vartriangle F_2$ there exists $y \in F_1 \vartriangle F_2$ such that $F_1 \vartriangle \{x, y\} \in \mathscr{F}$.

By (SEA), it is easy to see that matroids are the delta matroids having equicardinal sets for $\mathscr{F}$, and that $\mathscr{H} = \mathrm{Max}\{X \subseteq V : X \in \mathscr{F}\}$ and $\mathscr{B} = \mathrm{Min}\{X \subseteq V : X \in \mathscr{F}\}$ are collections of bases of matroids, called respectively the *upper matroid* and the *lower matroid* of δ.

For $A \subseteq V$, $\mathscr{F} \vartriangle A = \{F \vartriangle A : F \in \mathscr{F}\}$ is the base set of a delta matroid on V, denoted by $\delta \vartriangle A$. This new delta matroid, got by *twisting*, is said to be *equivalent to* δ. The special case $\delta^* = \delta \vartriangle V$ is called the *dual* of δ.

The next structure can be considered as an axiomatization of the concept of the class of equivalent delta matroids.

Symmetric Matroids

A *symmetric set* is a finite set W that can be expressed as $W = V \cup V^{\sim}$, where $V^{\sim}$ is a disjoint copy of V. Thus we have an *involution without a fixed point*, denoted $x \mapsto x^{\sim}$, of W into W. An equivalent way of looking at it is to say that W is partitioned by pairs $\{x, x^{\sim}\}$. A *transversal* (respectively, a *subtransversal*) of W is any subset $T \subseteq W$ satisfying $|T \cap \{x, x^{\sim}\}| = 1$ (respectively, $|T \cap \{x, x^{\sim}\}| \leqslant 1$) for all $x \in W$. Let $T(W)$ be the set of the subtransversals of W. Then $X, Y \subseteq W$ are said to be *compatible* if $X \cup Y \in T(W)$.

Let $\mathscr{T}$ be a nonempty set of transversals, which we call *bases*. $S \in T(W)$ is an *independent set* if there exists a base containing X, and it is a *circuit* if it is a minimal nonindependent set.

A symmetric matroid $\mathfrak{S}$ is a pair $\mathfrak{S} = (W, \mathcal{T})$, where W is a symmetric set and $\mathcal{T} \neq \emptyset$ is a set of bases, which satisfies the following axiom:

(SMA) For $T_1, T_2 \in \mathcal{T}$ and $x \in T_1 \triangle T_2$ there exists $y \in T_1 \triangle T_2$ such that $T_1 \triangle \{x, x^{\sim}, y, y^{\sim}\} \in \mathcal{T}$.

The concepts of duality and minors of matroids can be extended:

The *dual* of a symmetric matroid $\mathfrak{S} = (W, \mathcal{T})$ is $\mathfrak{S}^* = (W, \mathcal{T}^* = \{W \setminus T : T \in \mathcal{T}\})$, which is isomorphic to $\mathfrak{S}$ by the involution $x \mapsto x^{\sim}$.

For $A \in T(W)$ and $S = A \cup A^{\sim}$, we define

(a) the *restriction of* A: $\mathfrak{S} \setminus A = (W \setminus S, \mathcal{T} \setminus A = \{T \setminus S : T \in \mathcal{T}$ and $T \cap A$ is minimal$\})$;

(b) the *contraction of* A: $\mathfrak{S}/A = (W \setminus S, \mathcal{T}/A = \{T \setminus S : T \in \mathcal{T}$ and $T \cap A$ is maximal$\})$;

(c) the *projection on* S: $\mathfrak{S}(S) = (S, \mathcal{T}(S) = \{T \cap S \mid T \in \mathcal{T}\})$.

We can verify that $\mathfrak{S} \setminus A$, $\mathfrak{S}/A$, and $\mathfrak{S}(S)$ are symmetric matroids [6, 7]; they are called the *principal minors* of $\mathfrak{S}$. A *minor* (respectively, a *strong minor*) is the result of a sequence of restrictions, contractions, or projections (respectively, restrictions or contractions).

Lastly, note that a symmetric matroid can also be defined by its circuits as follows [7]:

THEOREM 1.1. *$\mathcal{C} \subseteq T(W)$ is the circuit set of a symmetric matroid on W if and only if:*

(C1) $\emptyset \notin \mathcal{C}$.

(C2) *If $C_1, C_2 \in \mathcal{C}$, then $|C_1 \cap C_2^{\sim}| \neq 1$.*

(C3) *If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.*

(C4) *For C_1 and C_2 distinct and compatible members of $\mathcal{C}$, and $x \in C_1 \cap C_2$, there exists $C \in \mathcal{C}$ such that $C \subseteq C_1 \cup C_2 - x$.*

For $A \in T(W)$, these axioms show that the family of independent sets included in A is the family of independent sets of a matroid, denoted $\mathcal{M}(\mathfrak{S}, A)$.

Correspondence of Structures

With any delta matroid $\delta = (V, \mathcal{F})$, we associate a symmetric matroid $\mathfrak{S} = (W, \mathcal{T})$ on the symmetric set $W = V \cup V^{\sim}$, with base set $\mathcal{T} = \{F \cup (V \setminus F)^{\sim} : F \in \mathcal{F}\}$. Conversely, with any symmetric matroid $\mathfrak{S} = (W, \mathcal{T})$ and transversal V of W, we associate a delta matroid $\delta = (V, \mathcal{F} = \{T \cap V : T \in \mathcal{T}\})$, called the *trace of* $\mathfrak{S}$ *on* V. For every other transversal T, the trace of $\mathfrak{S}$ on T is isomorphic to some equivalent of δ, and any equivalent of δ is isomorphic to some trace of $\mathfrak{S}$. Thus, a symmetric matroid can be seen as an equivalence class of delta matroids.

The concepts of circuits, independent sets, and minors of a symmetric matroid $\mathfrak{S}$ have an immediate interpretation in any trace δ of $\mathfrak{S}$. Let $D(V) := \{(X,Y): X, Y \subseteq V, X \cap Y = \emptyset\}$. Then $(X,Y) \in D(V)$ is a circuit (independent set) of δ if and only if $X \cup Y^{\sim}$ is a circuit (independent set) of $\mathfrak{S}$. The principal minors of δ are traces of principal minors of $\mathfrak{S}$. Thus for $A \subseteq V$, we have

(a) the *restriction* $\delta \setminus A = (V \setminus A, \mathscr{F} \setminus A = \{F \setminus A : F \in \mathscr{F}$ and $F \cap A$ is minimal$\})$.

(b) the *contraction* $\delta/A = (V \setminus A, \mathscr{F}/A = \{F \setminus A : F \in \mathscr{F}$ and $F \cap A$ is maximal$\})$.

(c) the *projection* $\delta(A) = (A, \mathscr{F}(A) = \{F \cap A \mid F \in \mathscr{F}\})$.

The properties of these minors are similar to those of the minors of matroids and are used without a proof.

Full Delta Matroids

Let $\mathfrak{U}$, $\mathfrak{L}$ be two matroids on V such that $\mathfrak{L}$ is a *quotient* (or *contraction*) of $\mathfrak{U}$ (see [14, Chapter 17.4]), and let $\mathscr{H}$, $\mathscr{B}$ be their sets of bases, respectively. Then $\delta = (V, \mathscr{F})$, where $\mathscr{F} = \{X \subseteq V : \exists H \in \mathscr{H}, \exists B \in \mathscr{B}$ with $B \subseteq X \subseteq H\}$, is a delta matroid, which is said to be *full*. These delta matroids are the *g-matroids* of E. Tardos [12]. Here is one of their characterizations, by circuits [8]:

THEOREM 1.2. *δ is full if and only if for any circuit (P,Q), $P = \emptyset$ or $Q = \emptyset$.*

The full delta matroids give an interesting example of a generalization of Edmond's intersection theorem; also, the two classes defined in the continuation of this paper, and including these delta matroids, will allow perhaps the extension of this theorem.

2. SYMMETRIC MATROIDS WITH WEAKLY SEPARATE CIRCUITS

Let $\mathfrak{S} = (W, \mathscr{T})$ be a symmetric matroid on W with its set of circuits $\mathscr{C}$. Consider the simple graph G on the vertex set W whose edge set is $\{xy : x \neq y$ and $x, y \in C$ for some $C \in \mathscr{C}\}$. $\mathscr{C}$ is said to be *weakly separate*, and $\mathfrak{S}$ is *transitive*, if all connected components of G are subtransversals of

W. By definition of $\mathfrak{S}^*$, it is clear that this property is preserved by duality. The following proposition justifies the name "transitive."

THEOREM 2.1. *$\mathfrak{S}$ is transitive if and only if one of the following equivalent statements holds*:

(1) *All connected components of G are cliques.*
(2) *If $C_1, C_2 \in \mathscr{C}$ and $C_1 \cap C_2 \neq \varnothing$ then $C_1 \cap C_2^{\sim} = \varnothing$.*

Proof. Suppose that $\mathfrak{S}$ is transitive, and consider a connected component K of G. Since K is a subtransversal, we can consider the matroid $M = \mathscr{M}(\mathfrak{S}, K)$, which is connected. So, for $x, y \in K$, $x \neq y$, there exists a circuit C of M, and hence of $\mathfrak{S}$, such that $x, y \in C$. This shows that K is a clique.

(1) $\Rightarrow$ (2): Let us assume that there exist $C_1, C_2 \in \mathscr{C}$ with $C_1 \cap C_2 \neq \varnothing$ and $C_1 \cap C_2^{\sim} \neq \varnothing$. Then for $x \in C_1 \cap C_2^{\sim}$, x and $x^{\sim}$ belong to the same connected component K of G, and $xx^{\sim} \notin G$, since the circuits of $\mathfrak{S}$ are subtransversals. This contradicts (1).

(2) $\Rightarrow$ $\mathfrak{S}$ is transitive: Let K be a connected component of G. If $K \notin T(W)$, there exist $x, x^{\sim} \in K$ and a sequence $\mu = (C_i : i = 1, \dots, n)$ of circuits of $\mathfrak{S}$ such that $x \in C_1$, $x^{\sim} \in C_n$, and $C_i \cap C_{i+1} \neq \varnothing$. Choose those x, $x^{\sim}$, and μ with n minimum. We have $n \geqslant 2$ and more exactly $n = 2$. Indeed, by choice of n, we have $A = \bigcup(C_i : i = 1, \dots, n-1) \in T(W)$, and so the matroid $M = \mathscr{M}(\mathfrak{S}, A)$ is connected. Consequently, some $a \in C_{n-1} \cap C_n$ is connected to x in M and there exists a circuit C of $\mathfrak{S}$ such that $a, x \in C$. Now, C and C_n contradict (2). ∎

By considering the families of circuits of various principal minors [7], we deduce easily from Theorem 2.1(2):

COROLLARY 2.2. *$\mathfrak{S}$ is transitive if and only if any minor is transitive.*

Let H be a class of symmetric matroids, closed by minors. We define an *obstruction* (respectively, a *strong obstruction*), with respect to H, as a symmetric matroid that does not belong to H and is minimal by (principal) minors (respectively, strong minors) for this property.

When H is the class of transitive symmetric matroids, these concepts are equivalent, as we can see by the following lemma.

LEMMA 2.3. *Let $\mathfrak{S}(W)$ be a strong obstruction with respect to the transitive symmetric matroids. For any pair $\{C_1, C_2\}$ of circuits of $\mathfrak{S}(W)$*

that does not satisfy (2), *we have* $|C_1 \cap C_2| = 1$, $W = C_1 \cup C_1^{\sim}$, *and* C_1, C_2 *are the only circuits included in* $C_1 \cup C_2$.

Proof. If $\mathfrak{S}$, defined on W, is a strong obstruction, by Theorem 2.1(2) there exist circuits C_1, C_2 such that $C_1 \cap C_2 \neq \emptyset$ and $C_1 \cap C_2^{\sim} \neq \emptyset$.

Claim 1: $|C_1 \cap C_2| = 1$. Let $a \in C_1 \cap C_2$, and consider $A = (C_1 \cap C_2) - a$. We have $A \in T(W)$, and for $i = 1, 2$, $C_i' = C_i \setminus A$ are circuits of $\mathfrak{S}/A$ that do not satisfy (2). Hence by minimality of $\mathfrak{S}$, we have $A = \emptyset$.

Now we consider $X = C_1 \cap C_2^{\sim}$, $x \in X$, $B = (C_1 \cup C_2) \setminus (X^{\sim} + x)$, and $B' = (C_1 \cup C_2) \setminus (X + x^{\sim})$.

Claim 2: *B and B′ are independent sets.* Since $B \in T(W)$, consider the matroid $M = \mathcal{M}(\mathfrak{S}, B)$, and let D be a base of M such that $C_1 - x \subseteq D$. If $D \neq B$, then for some $b \in B \setminus D$, there exists a circuit C_3 such that $C_3 \setminus D = b$. Necessarily $C_3 - C_2 \neq \emptyset$, and for some $y \in C_3 - C_2$ we have $y \in C_1 \cap C_3$. Hence, by the elimination axiom (C4), which also holds in matroids, there exists a circuit C_4 satisfying $x \in C_4 \subseteq C_1 \cup C_3 - y$, and thus $b \in C_4$. Since $\{C_2, C_4\}$ does not satisfy (2), by Claim 1 we have $a \notin C_4$. Let us suppose that there exists a circuit S of $\mathfrak{S}$ such that $a \in S$ and $\emptyset \neq S - a \subseteq C_4$. Then we have $S \subseteq C_1 \cup C_3$ and $y \notin S$. Since $\{C_1, C_3\}$ is a modular pair of distinct circuits (see [13]), by uniqueness we obtain $S = C_4$ and so $a \in C_4$, which is a contradiction. Therefore, $\{C_2 - a, C_4\}$ is a pair of circuits of $\mathfrak{S}/a$ that does not satisfy (2), in contradiction with the minimality of $\mathfrak{S}$. Hence, B is an independent set of $\mathfrak{S}$, and similarly for B'.

Consider $I = (C_1 \cup C_2) \setminus (X \cup X^{\sim})$.

Claim 3: $I = \{a\}$. Indeed, denote $A = I - a$ and $\mathfrak{S}' = \mathfrak{S}/A$. Since B, B' are independent sets of $\mathfrak{S}$ (by Claim 2), $C_i' = C_i \setminus A$ are circuits of $\mathfrak{S}'$ for $i = 1, 2$ that do not satisfy (2). Hence $A = \emptyset$, by minimality of $\mathfrak{S}$.

Claim 4: $W = C_1 \cup C_1^{\sim}$. Else, consider $c \in W$ such that $c \notin C_1 \cup C_1^{\sim}$. Assume that there exists a circuit S of $\mathfrak{S}$ satisfying $c \in S$ and $\emptyset \neq S - c \subseteq C_1$. If $a \in S$, we have $S = \{a, c\}$, else $\{S, C_2\}$ does not satisfy (2) (by Claim 3) and so contradicts Claim 3. Hence, for $i = 1, 2$, $C_i' = C_i - a + c$ is a circuit in $\mathfrak{S}$. Now, by axiom (C2) of Theorem 1.1, C_i' is a circuit of $\mathfrak{S}/a^{\sim}$, and since $\{C_1', C_2'\}$ opposes (2), the minimality of $\mathfrak{S}$ is contradicted. So, we have $a \notin S$. Since $S \neq \{c\}$ and S, C_1 are compatible, there exist $z \in S \cap C_1$ and a circuit S' satisfying $a \in S' \subseteq S \cup C_1 - z$ with, moreover, $c \in S'$. Hence a contradiction, by considering S' in place of S. So C_1 is a circuit of $\mathfrak{S}/c$, and similarly C_2. But this contradicts the minimality of $\mathfrak{S}$.

By the previous claims, we can now see that C_1, C_2 are the only circuits included in $C_1 \cup C_2$. ∎

REMARK. The above proof can be simplified if we only suppose that $\mathfrak{S}$ is an obstruction (and not a strong obstruction).

Now, the characterization of obstructions to transitive symmetric matroids is easy. We obtain:

THEOREM 2.4. *$\mathfrak{S}$ is a (strong) obstruction with respect to transitive symmetric matroids if and only if it is isomorphic to one of the following symmetric matroids, defined on W by their set of circuits*:

Type 1. $W = X \cup X^{\sim} \cup \{a, a^{\sim}\}$, *with* $X \cap \{a, a^{\sim}\} = \varnothing$, $|X| \geqslant 2$, *and*

$$\mathscr{C} = \{aX, aX^{\sim}\}.$$

Type 2. $W = \{a, b, c, a^{\sim}, b^{\sim}, c^{\sim}\}$, *and*

$$\mathscr{C} = \{abc, ab^{\sim}c^{\sim}, ba^{\sim}c^{\sim}\} \quad \textit{or} \quad \mathscr{C} = \{abc, ab^{\sim}c^{\sim}, ba^{\sim}c^{\sim}, ca^{\sim}b^{\sim}\}.$$

Proof. Let $\mathfrak{S}(W)$ be isomorphic to one of these symmetric matroids. By Theorem 2.1(2), $\mathfrak{S}(W)$ is not transitive and the strong minimality is immediate.

Conversely, let $\mathfrak{S}(W)$ be a strong obstruction. By Theorem 2.1(2) and Lemma 2.3, there exist two distinct circuits C_1, C_2 such that $|C_1 \cap C_2| = 1$ and $W = C_1 \cap C_1^{\sim} = C_2 \cup C_2^{\sim}$. Consider $a = C_1 \cap C_2$ and $X = C_1 - a$. Necessarily, we have $X^{\sim} = C_2 - a$ and, since $|C_1 \cap C_2^{\sim}| \neq 1$, $|X| \geqslant 2$.

If no circuit contains $a^{\sim}$, by Lemma 2.3 we have $\mathscr{C} = \{C_1, C_2\}$, and so we obtain one of the symmetric matroids of type 1.

Else, there exists a circuit C with $a^{\sim} \in C$ and $C \neq \{a^{\sim}\}$ [by axiom (C2)]. Hence for example, we have $C \cap C_1 \neq \varnothing$. Then $\{C, C_1\}$ does not satisfy Theorem 2.1(2), and by Lemma 2.3 we get $|C \cap C_1| = 1$ and $W = C \cup C^{\sim}$. Therefore, we have $X = \{b, c\}$ with necessarily $C = \{b, a^{\sim}, c^{\sim}\}$ or $C = \{c, a^{\sim}, b^{\sim}\}$ to satisfy axiom (C2). Hence, we obtain one of the symmetric matroids of type 2. ∎

3. SYMMETRIC MATROIDS WITH SEPARATE CIRCUITS

Let $\mathfrak{S} = (W, \mathscr{T})$ be a symmetric matroid on W, and $\mathscr{C}$ the set of its circuits. $\mathscr{C}$ is said to be *separate* and $\mathfrak{S}$ to be *simple* if there exists a transversal V of W such that

(3) For any circuit C we have $C \subseteq V$ or $C \subseteq V^{\sim}$.

Obviously we have the proposition "simple $\Rightarrow$ transitive," the converse being false. Moreover, it is clear that this property is preserved by duality. By considering the family of circuits of various principal minors, we easily deduce

PROPOSITION 3.1. *$\mathfrak{S}$ is simple if and only if any minor is simple.*

Return now to the graph G associated to $\mathfrak{S}$. Let $\mathscr{K}$ be the set of connected components of G, and consider the simple graph Γ on the vertex set $\mathscr{K}$ whose edge set is $\{K_1K_2 : K_1 \neq K_2,\ |K_i| \geqslant 2, \text{ and } K_1 \cap K_2^{\sim} \neq \varnothing\}$. We notice that $K_1K_2 \in \Gamma$ if and only if there exist circuits C_i of $\mathfrak{S}$ such that $C_i \subseteq K_i$ for $i = 1, 2$, with $C_1 \cap C_2^{\sim} \neq \varnothing$. The graph Γ characterizes the simple symmetric matroids.

THEOREM 3.2. *$\mathfrak{S}$ is simple if and only if $\mathfrak{S}$ is transitive and Γ is bipartite.*

Proof. First, consider the simple graph Γ' on the vertex set $\mathscr{K}$ whose edge set is $\{K_1K_2 : K_1 \neq K_2 \text{ and } K_1 \cap K_2^{\sim} \neq \varnothing\}$, and note that for $K \in \mathscr{K}$ with $|K| = 1$, the degree of K in Γ' is one. Therefore, Γ is bipartite if and only if Γ' is bipartite.

If $\mathfrak{S}$ is simple, then $\mathfrak{S}$ is transitive and, by (3), Γ is bipartite with $\{K \in \mathscr{K} : K \subseteq V\}$ and $\{K \in \mathscr{K} : K \subseteq V^{\sim}\}$ as color classes.

Conversely, suppose that $\mathfrak{S}$ is transitive and Γ is bipartite. By the above remark, Γ' is bipartite. Consider a color class $\mathscr{E}$ of Γ', and let T be the set of vertices of W corresponding to $\mathscr{E}$. We have $T \in T(W)$. Else, there exist distinct $K_1, K_2 \in \mathscr{E}$ and some $x \in T \cap T^{\sim}$ such that $x \in K_1 \cap K_2^{\sim}$. Hence we have $K_1K_2 \in \Gamma'$, which contradicts $K_1, K_2 \in \mathscr{E}$. For $X \subseteq W$, denote by $\overline{X}$ the union of elements of $\mathscr{K}$ that intersect X. Note that $T = \overline{T}$, and show that T is a transversal of W. Else, there exist $x, x^{\sim} \in W \setminus T$ and $K_1, K_2 \in \mathscr{K}$ such that $x \in K_1$, $x^{\sim} \in K_2$. Since $\mathfrak{S}$ is transitive and $T = \overline{T}$, we have $K_1 \neq K_2$ and $K_i \cap T = \varnothing$ for $i = 1, 2$. But $x \in K_1 \cap K_2^{\sim}$ implies $K_1K_2 \in \Gamma'$, and so, by the choice of T, we have $K_1 \subseteq T$ or $K_2 \subseteq T$, which is a contradiction. Now since $T = \overline{T}$, for any $K \in \mathscr{K}$ we have $K \subseteq T$ or $K \subseteq T^{\sim} = W \setminus T$. Therefore (3) is true and $\mathfrak{S}$ is simple. ■

Consider an integer $n \geqslant 2$ and a set V with a partition $(E_i : i = 0, \ldots, n-1)$ such that $|E_i| \geqslant 2$ for any i. Denote by $R_n(E_i : i = 0, \ldots, n-1)$, or simply R_n, the symmetric matroid on $W = V \cup V^{\sim}$ defined by the set of circuits (the subscripts being taken modulo n):

$$\mathscr{C}_n = \{C_1 = E_i \cup E_{i-1}^{\sim} : i = 0, \ldots, n-1\}.$$

For any $n \geqslant 2$, it is easy to verify that R_n is transitive and, by Theorem 3.2 when n is odd, that R_n is not simple and is minimal (by minors). In fact, these symmetric matroids are the only transitive obstructions with respect to simple symmetric matroids, as the following lemma proves.

LEMMA 3.3. *Any transitive (strong) obstruction, with respect to the simple symmetric matroids, is isomorphic to some* R_{2p+1}.

Proof. Let $\mathfrak{S}(W)$ be a transitive strong obstruction, and consider the associated graphs G and Γ, which by Theorem 3.2 is not bipartite. So there exists an elementary odd cycle $\mu = (K_0, \dots, K_{n-1})$, where $n = 2p + 1 \geqslant 3$, and the K_i are distinct connected components of G such that $K_i \cap K_{i+1}^{\sim} \neq \varnothing$ and $|K_i| \geqslant 2$ for $i = 0, \dots, n-1$ (modulo n). Choose such a cycle with n minimum. Then $K_i \cap K_j^{\sim} = \varnothing$ is satisfied, for $j \neq i-1, i+1$, and there exists a sequence $(x_0, y_0, \dots, x_{n-1}, y_{n-1})$ such that $x_i \neq y_i = x_{i+1}^{\sim}$, and $x_i, y_i \in K_i$ for $i = 0, \dots, n-1$ (modulo n).

Now, $\mathfrak{S}$ being transitive, for each i we can choose circuits $C_i \subseteq K_i$ such that $x_i, y_i \in C_i$. Then we have $y_i \in E_i = C_i \cap C_{i+1}^{\sim} \neq \varnothing$ with $|E_i| \geqslant 2$, by axiom (C2) of Theorem 1.1, and moreover, $E_i \cap E_j = \varnothing$ for $i \neq j$.

Consider the sets $A = \bigcup_{i=0}^{n-1} E_i$ and $B = (\bigcup_{i=0}^{n-1} C_i) \setminus (A \cup A^{\sim})$.

Claim 1: $B = \varnothing$ *and* $C_i = E_i \cup E_{i-1}^{\sim}$ *for i modulo n.* Firstly, we have $B \in T(W)$, else there exist $x \in B$ and C_i, C_j, $i \neq j$, such that $x \in C_i \cap C_j^{\sim}$. Then, by the choice of μ and C_k, we have $j = i \pm 1$ (modulo n). Hence $x \in A \cup A^{\sim}$, which contradicts the definition of B.

Now since $B \cap C_i^{\sim} = \varnothing$, we have $B \cup C_i \in T(W)$ for $i = 0, \dots, n-1$. Let $D_i = B \cup (C_i - y_i)$, and consider $\mathfrak{S}' = \mathfrak{S}/B$. Since the K_j are distinct connected components, D_i is an independent set of $\mathfrak{S}$, and so $C_i - B = E_i \cup E_{i-1}^{\sim}$ is a circuit of $\mathfrak{S}'$ for $i = 0, \dots, n-1$. But then by Theorem 3.2, $\mathfrak{S}'$ is not simple, and so, by minimality of $\mathfrak{S}$, we have $B = \varnothing$, and Claim 1 holds.

Claim 2: $W = \bigcup_{i=0}^{n-1} K_i$, *and* $K_i = C_i$ *for* $i = 0, \dots, n-1$.

(a) Suppose that there exists $b \notin K_j \cup C_j^{\sim}$. Then C_j is a circuit of $\mathfrak{S}/b$. Else, there exists a circuit S of $\mathfrak{S}$ such that $b \in S$ and $\varnothing \neq S - b \subseteq C_j$, and hence by connectivity, we have $b \in K_j$, which is a contradiction. By Claim 1, for $X = \bigcup_{i=0}^{n-1} K_i$, note that we have $\bigcup_{i=0}^{n-1} C_i^{\sim} = \bigcup_{i=0}^{n-1} C_i \subseteq X$. Therefore $W = X$ is true, else by the above argument and Theorem 3.2, $\mathfrak{S}/b$ is not simple for $b \in W \setminus X$, which contradicts the minimality of $\mathfrak{S}$. Now, for a given subscript i, we prove $K_i = C_i$. Else, we consider $b \in K_i \setminus C_i$. $\mathfrak{S}$ being transitive, denote by M_i the matroid $\mathscr{M}(\mathfrak{S}, K_i)$. By definition of K_i, M_i is connected. Hence, we deduce that there exists a circuit S in $\mathfrak{S}$ such that $\{S, C_i\}$ is a modular pair of distinct circuits of M_i, with $b \in S$ and

$\emptyset \neq S - b \subseteq C_i$. Since $W = X$ and by the choice of μ, we have necessarily $b^{\sim} \in K_j \setminus C_j$ for some $j \in \{i+1, i-1\}$. Exchanging b and $b^{\sim}$ if necessary, we can suppose $j = i + 1$. Therefore, as before, there exists a circuit D in $\mathfrak{S}$ such that $\{D, C_{i+1}\}$ is a modular pair of distinct circuits of M_{i+1}, with $b^{\sim} \in D$ and $\emptyset \neq D - b^{\sim} \subseteq C_{i+1}$.

(b) We can assume that $S \cap E_{i-1}^{\sim} \neq \emptyset$ and $S \cap E_i \neq \emptyset$. Indeed, if $S \cap E_{i-1}^{\sim} = \emptyset$, consider $x \in S \cap E_i \neq \emptyset$ and the circuit S' of M_i satisfying $S' \subseteq S \cup C_i - x$. Then we have $E_{i-1}^{\sim} + b \subseteq S'$, and $E_i \cap S' \neq \emptyset$, else $S' \cap D^{\sim} = b$ contradicts axiom (C2) of Theorem 1.1. So we can exchange S, S' if necessary. Similarly, $S \cap E_i = \emptyset$ implies the contradiction $S \cap D^{\sim} = b$.

(c) Now, we consider the families of circuits $\mathfrak{U} = (C_j : j = 0, \ldots, n-1)$ and $\mathfrak{U}' = \mathfrak{U} - C_i + S$. Since we have $C_j' \subseteq K_j$ and $E_j' = C_j' \cap C_{j+1}'^{\sim} \neq \emptyset$, Claim 1 applied to $\mathfrak{U}'$ yields $C_j' = E_j' \cup E_{j-1}'^{\sim}$ for j modulo n. When $j = i$, $i - 1$, since $C_i' = S$ and $C_k' = C_k$ for $k \neq i$, we obtain $E_i' = S \cap C_{i+1}^{\sim}$, $E_{i-1}' = C_{i-1} \cap S^{\sim}$, $E_{i+1}' = E_{i+1}$, $E_{i-2}' = E_{i-2}$. But by Claim 1, we have $C_i - 1 = E_{i-1}' \cap E_{i-2}'^{\sim} = E_{i-1} \cup E_{i-2}^{\sim}$ and $C_{i+1} = E_{i+1}' \cup E_i'^{\sim} = E_{i+1} \cup E_i^{\sim}$. Hence we deduce $E_{i-1}' = E_{i-1}$ and $E_i' = E_i$.

So the equations $E_i = S \cap C_{i+1}^{\sim} = S \cap (E_{i+1}^{\sim} \cup E_i)$ and $E_{i-1}^{\sim} = C_{i-1}^{\sim} \cap S = S \cap (E_{i-1}^{\sim} \cup E_{i-2})$ follow, with the consequences $E_i \subseteq S$ and $E_{i-1}^{\sim} \subseteq S$, which yield $C_i \subseteq S$, which is absurd.

So we have $K_i \setminus C_i = \emptyset$, and Claim 2 is true.

Lastly, by Claim 2, $\mathscr{C} = (C_i : i = 0, \ldots, n-1)$ is the family of circuits of $\mathfrak{S}$. Hence by Claim 1, $\mathscr{C} = \{E_i \cup E_{i-1}^{\sim} : i = 0, \ldots, n-1\}$, and $\mathfrak{S}$ is isomorphic to some R_n, for some odd integer $n \geqslant 3$. ∎

Notice again that the condition "strong" makes the proof of the above Claim 2 much more complicated.

Now, the characterization of simple symmetric matroids is deduced directly from Theorem 3.2, Lemma 3.3, and Theorem 2.4:

THEOREM 3.4. *$\mathfrak{S}$ is a (strong) obstruction with respect to simple symmetric matroids if and only if it is isomorphic to one of Type* 1 *or Type* 2, *or to some R_{2p+1} with $p \geqslant 1$.*

4. CHARACTERIZATION OF TWO CLASSES OF DELTA MATROIDS COMING FROM MATROIDS

Let $M(V)$ be a matroid on V with the base set $\mathscr{B}$ and the family of independent sets $\mathscr{I}$. (V, B) and $(V, \mathscr{I})$ are two particular cases of full delta matroids, called *b-matroids* and *i-matroids*, respectively.

Theorems 2.4 and 3.4 allow to characterize their equivalence classes. A delta matroid $\delta = (V, \mathcal{F})$ is said to be *even* if for any $F_1, F_2 \in \mathcal{F}$, $F_1 \vartriangle F_2$ has even cardinality. This property is preserved by strong minors (but not by projection), and since it is also preserved by equivalence, its associated symmetric matroid is also said to be even. We show easily that there exists only one obstruction.

PROPOSITION 4.1. *A delta matroid is even if and only if it does not contain strong minors isomorphic to* $\delta_0 = (\{a\}, \{\varnothing, a\})$.

PROPOSITION 4.2. *Let* δ *be an even delta matroid, and let* $\mathfrak{S}$ *be its associated symmetric matroid. Then the following properties are equivalent.*

(i) δ *(or* $\mathfrak{S}$*) is transitive.*
(ii) δ *(or* $\mathfrak{S}$*) is simple.*
(iii) δ *is equivalent to a b-matroid.*
(iv) δ *does not contain strong minors isomorphic to*

$$\delta_1 = (\{a, b, c\}, \{\varnothing, ab, bc, ca\}) \quad \textit{or} \quad \delta_2 = \delta_1^* = (\{a, b, c\}, \{a, b, c, abc\})$$

Proof. Since any matroid is a full delta matroid, Theorem 1.2 yields (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iv): By Theorem 2.4, the only even obstruction for the property to be transitive is the symmetric matroid $\mathfrak{S}$ on $W = \{a, b, c, a^{\sim}, b^{\sim}, c^{\sim}\}$ defined by the circuit set $\mathscr{C} = \{abc, ab^{\sim}c^{\sim}, ba^{\sim}c^{\sim}, ca^{\sim}b^{\sim}\}$. By Proposition 4.1, we obtain the equivalent delta matroids δ_1 and δ_2.

(iv) $\Rightarrow$ (iii): Suppose (iv) holds and δ is even. By Theorem 3.4 and Proposition 4.1, δ is simple, since the symmetric matroids of type 1 and the R_n are not even. So by (1.2) there exists $A \subseteq V$ such that $\delta \vartriangle A$ is full. Since $\delta \vartriangle A$ is even, it is necessarily a matroid. Hence (iii) holds. ■

From Proposition 4.1 and Proposition 4.2, we get directly

COROLLARY 4.3. δ *is equivalent to a b-matroid if and only if it does not contain strong minors isomorphic to* δ_0 *or* δ_1 *or* δ_2.

Consider a set X with $|X| \geqslant 2$ and $\mathscr{C} = \{X, X^{\sim}\}$. It is clear that $\mathscr{C}$ is a family of circuits of a simple symmetric matroid on $X \cup X^{\sim}$, which is said to be of *type* 3. These symmetric matroids are obstructions for the symmetric matroids associated with i-matroids.

PROPOSITION 4.4. *Let $\delta(V)$ be a delta matroid with its associated symmetric matroid $\mathfrak{S}(W)$. Then the following properties are equivalent:*

(i) *δ is equivalent to an i-matroid.*
(ii) *Some transversal T of W contains any circuit of $\mathfrak{S}$.*
(iii) *For all circuits C_1, C_2 of $\mathfrak{S}$, we have $C_1 \cap C_2^{\sim} = \varnothing$.*
(iv) *Any minor of δ is equivalent to an i-matroid.*
(v) *$\mathfrak{S}$ is transitive and does not contain (strong) minors of type* 3.

Proof. Since a symmetric matroid associated to an i-matroid has the same circuit set as the matroid, we have easily (iv) $\Rightarrow$ (i) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii): Suppose (iii), and let X be the union of all circuits of $\mathfrak{S}$. We deduce from (iii) that X is a subtransversal. So there exists a transversal T of W such that $X \subseteq T$, and (ii) follows.

(ii) $\Rightarrow$ (iv) results from the description of circuit sets of principal minors.

(iii) $\Rightarrow$ (v) follows from Theorem 2.1(2) and (iv), since the symmetric matroids of type 3 do not satisfy (iii).

(v) $\Rightarrow$ (iii): Let $\mathfrak{S}(W)$ be a transitive symmetric matroid which does not satisfy (iii), and minimal by strong minors. Choose circuits C_1, C_2 of $\mathfrak{S}$ with $X = C_1 \cap C_2^{\sim} \neq \varnothing$ such that $|X|$ is as small as possible. Consider $A = (C_1 \cup C_2) \setminus (X \cup X^{\sim})$. Since $\mathfrak{S}$ is transitive, C_1 and C_2 belong to distinct components. Therefore, $X = C_1 \setminus A$ and $X^{\sim} = C_2 \setminus A$ are circuits of $\mathfrak{S}/A$. By minimality of $\mathfrak{S}$, we have $A = \varnothing$. We show that $W = X \cup X^{\sim}$ is satisfied. Else, let $b \in W \setminus (X \cup X^{\sim})$. If there exists a circuit S such that $b \in S$ and $\varnothing \neq S - b \subseteq X$, then we have $C_2^{\sim} \cap S \subseteq X$, and by the choice of X, $C_2^{\sim} \cap S = X$. Hence, we have $C_1 = X \subseteq S - b$, which is a contradiction. So X is a circuit of $\mathfrak{S}/b$, and similarly $X^{\sim}$. But this contradicts the minimality of $\mathfrak{S}$. Now since $\mathfrak{S}$ is transitive, it is clear that C_1 and C_2 are the only circuits, and so $\mathfrak{S}$ is of type 3. ■

5. OTHER INTERPRETATION FOR DELTA MATROIDS

Let $\delta = (V, \mathcal{F})$ be a delta matroid with its associated symmetric matroid $\mathfrak{S} = (W, \mathcal{T})$ on the symmetric set $W = V \cup V^{\sim}$. By (3), Theorem 1.2, and the interpretation of circuits of δ in $\mathfrak{S}$, we see that $\mathfrak{S}$ is simple if and only if δ is equivalent to some full delta matroid. From this remark and Theorem 3.4, we could deduce a characterization of the class of those delta matroids, by excluded minors. Instead of that, we give another, more direct one, which will allow us also to deduce all full delta matroids that are equivalent to a given full delta matroid.

Consider the multigraph $G(\delta)$ on the vertex set V whose edges are covered by the following bicoloration $(R(\delta), B(\delta))$:

(4) $ab \in R(\delta) \Leftrightarrow a \neq b$ and there exists a circuit (P, Q) such that

$$(a \in P \text{ and } b \in Q) \quad \text{or} \quad (b \in P \text{ and } a \in Q);$$

$ab \in B(\delta) \Leftrightarrow a \neq b$ and there exists a circuit (P, Q) such that

$$a, b \in P \quad \text{or} \quad a, b \in Q.$$

Notice that we can obtain a double edge ab if $ab \in R(\delta) \cap B(\delta)$. Obviously, there exist some relations between $G(\delta)$ and the graph G defined at the beginning of Section 2. For example, we have

PROPOSITION 5.1.

(1) *δ is full* $\Leftrightarrow R(\delta) = \varnothing$.
(2) *$\mathfrak{S}$ is transitive* $\Leftrightarrow R(\delta) \cap B(\delta) = \varnothing$.

Proof. (1): It is a direct consequent of (4) and Theorem 1.2.

(2): Assume $ab \in R(\delta) \cap B(\delta)$. By the definitions (4), there exist circuits (P_i, Q_i), $i = 1, 2$, such that $a \in P_1$, $b \in Q_1$, and $a, b \in P_2$ or $a, b \in Q_2$. So the corresponding circuits C_i of $\mathfrak{S}$ are in the same connected component K of the graph G, and we have $b, b^{\sim} \in K$ or $a, a^{\sim} \in K$. Hence, $\mathfrak{S}$ is not transitive.

Conversely, if $\mathfrak{S}$ is not transitive, there exist circuits $C_i = (P_i, Q_i)$ of δ, $i = 1, 2$, satisfying $(P_1 \cap P_2) \cup (Q_1 \cap Q_2) \neq \varnothing$ with $C_1 \cap C_2^{\sim} \neq \varnothing$. Therefore we have, for example, $P_1 \cap P_2 \neq \varnothing$, and $P_1 \cap Q_2 \neq \varnothing$ or $P_2 \cap Q_1 \neq \varnothing$. Consider $a \in P_1 \cap P_2$. For $b \in (P_1 \cap Q_2) \cup (P_2 \cap Q_1)$, we have $ab \in R(\delta) \cap B(\delta)$. ∎

A *fundamental circuit with respect to* $F \in \mathcal{F}$ is any circuit (P, Q) of δ such that $|P - F| = 1$ and $Q \cap F = \varnothing$. Any circuit (P, Q) satisfying $P \neq \varnothing$ is a fundamental circuit with respect to some $F \in \mathcal{F}$. It is easy to prove [8]

LEMMA 5.2. *If* (P, Q) *is a fundamental circuit with respect to* $F \in \mathscr{F}$ *and if* $P \setminus F = x$, *then* $P - x = \{y \in V : F - y + x \in \mathscr{F}\}$ *and* $Q = \{y \in V : F + x + y \in \mathscr{F}\}$.

From this result and the dual result, we deduce immediately an equivalent definition of color classes $R(\delta)$ and $B(\delta)$ and their transformation by twisting. For $A \subseteq V$, let $\omega(A) := \{ab \in G(\delta) : a \in A,\ b \notin A\}$.

PROPOSITION 5.3. *For* $a, b \in V$, $a \neq b$, *we have*

(a) $ab \in R(\delta)$ $\Leftrightarrow$ *there exists* $F \in \mathscr{F}$, *with* $F \subseteq V - \{a, b\}$, *such that*

$$F \,\triangle\, \{a, b\} \in \mathscr{F} \quad \textit{and} \quad (F \,\triangle\, a \notin \mathscr{F} \text{ or } F \,\triangle\, b \notin \mathscr{F});$$

(b) $ab \in B(\delta)$ $\Leftrightarrow$ *there exists* $F \in \mathscr{F}$, *with* $a \in F$ *and* $b \notin F$, *such that*

$$F \,\triangle\, \{a, b\} \in \mathscr{F} \quad \textit{and} \quad (F \,\triangle\, a \notin \mathscr{F} \textit{ or } F \,\triangle\, b \notin \mathscr{F}).$$

Proof. Suppose $ab \in R(\delta)$. By (4) and without loss of generality, we can assume that $a \in P$ and $b \in Q$ for some circuit (P, Q). Hence there exists $F \in \mathscr{F}$ such that $F \subseteq V - \{a, b\}$, $P - F = \{a\}$, and $Q \cap F = \varnothing$. Therefore, we have $F \,\triangle\, a \notin \mathscr{F}$ and, by (5.2), $F \,\triangle\, \{a, b\} \in \mathscr{F}$. Conversely, if $F \in \mathscr{F}$ with $F \subseteq V - \{a, b\}$ satisfies $F \,\triangle\, \{a, b\} \in \mathscr{F}$ and, for example, $F \,\triangle\, a \notin \mathscr{F}$, then there exists a circuit (P, Q) such that $a \in P$ and, by Lemma 5.2, $b \in Q$. Hence $ab \in R(\delta)$ holds.

Similarly, we prove the second equivalence, using the duality if necessary. ∎

COROLLARY 5.4. *For* $A \subseteq V$, *we have*

$$R(\delta \,\triangle\, A) = R(\delta) \,\triangle\, \omega(A) \quad \textit{and} \quad B(\delta \,\triangle\, A) = B(\delta) \,\triangle\, \omega(A).$$

Proof. Since the base set of $\delta \,\triangle\, A$ is $\{F \,\triangle\, A \mid F \in \mathscr{F}\}$, the result follows directly from Proposition 5.3. ∎

Consider the multigraph $\Gamma(\delta)$ whose vertices are the connected components of the subgraph of $G(\delta)$ induced by $B(\delta)$, with the edge set corresponding to $R(\delta)$. More simply, $\Gamma(\delta)$ is the multigraph deduced from $G(\delta)$ by contraction of edges of $B(\delta)$. So $\Gamma(\delta)$ may contain loops. Now, we can simply characterize the full twistings of a full delta matroid and the case where $\mathfrak{S}$ is simple.

PROPOSITION 5.5.

(1) *If* δ *is full and* $A \subseteq V$, *then we have*

$$\delta \vartriangle A \textit{ is full} \quad \Leftrightarrow \quad A \textit{ is a union of connected components of } G(\delta).$$

(2) $\mathfrak{S}$ *is simple* $\Leftrightarrow$ $\Gamma(\delta)$ *is bipartite.*

Proof. (1): By Proposition 5.1, we have $R(\delta) = \emptyset$, and the equivalence of $\delta \vartriangle A$ is full with $R(\delta \vartriangle A) = \emptyset$, or again, by Corollary 5.4, with $\omega(A) = \emptyset$. Since $\omega(A) = \emptyset$ if and only if A is a union of connected components of $G(\delta)$, the result follows.

(2): It is clear that $\mathfrak{S}$ is simple if and only if for some $A \subseteq V$, $\delta \vartriangle A$ is full. Hence by Proposition 5.1, we have the equivalent property $R(\delta \vartriangle A) = \emptyset$, or again, by Corollary 5.4, $R(\delta) = \omega(A)$, which is equivalent to $\Gamma(\delta)$ bipartite. ■

6. SYMMETRIC MATROIDS CONSIDERED AS AN INTERSECTION

Let $\mathfrak{S} = (W, \mathcal{T})$ be a symmetric matroid on W and $\mathfrak{S}^*$ be its dual. For a given transversal V of W, we consider the matroids $M_1(V) = \mathcal{M}(\mathfrak{S}, V)$ (its circuits are the circuits C of $\mathfrak{S}$ with $C \subseteq V$), the dual $M_2(V)$ of $\mathcal{M}(\mathfrak{S}^*, V)$, and the delta matroid $\delta(V)$, trace of $\mathfrak{S}$ on V. By axiom (C2) of Theorem 1.1, we have the quotient of matroids $M_1(V) \to M_2(V)$ (see [14]). $M_1(V)$ and $M_2(V)$ are the upper matroid and the lower matroid of δ, respectively. Therefore, we have

(5) H a base of $M_1(V)$ $\Leftrightarrow$ $H = T \cap V$ with T a base of $\mathfrak{S}$ and $T \cap V$ maximal;
B a base of $M_2(V)$ $\Leftrightarrow$ $B = T \cap V$ with T a base of $\mathfrak{S}$ and $T \cap V$ minimal.

We denote by $\mathfrak{S}_V$ the symmetric matroid associated to the full delta matroid defined by this quotient. So $\mathfrak{S}_V$ is simple. Its circuits are the circuits C of $\mathfrak{S}$ such that $C \subseteq V$ or $C \subseteq V^{\sim}$, and its bases are the transversals T or W such that $B \subseteq T \cap V \subseteq H$ for some bases H of $M_1(V)$ and B of $M_2(V)$.

$\mathfrak{S}$ is said to be the *base intersection* of the symmetric matroids $\mathfrak{S}_1 = (W, \mathcal{T}_i)$, $i \in I$, if we have $\mathcal{T} = \bigcap(\mathcal{T}_i : i \in I)$. The following proposition asserts that any symmetric matroid is the base intersection of simple symmetric matroids.

PROPOSITION 6.1. *$\mathfrak{S}$ is the base intersection of $(\mathfrak{S}_V : V$ a transversal of $W)$.*

Proof. Let V be a given transversal of W, and $T \in \mathcal{T}$. By (5), there exists a base B of $M_2(V)$ and a base H of $M_1(V)$ with $B \subseteq T \cap V \subseteq H$. So T is a base of $\mathfrak{S}_V$. Conversely, if a transversal T of W is not a base of $\mathfrak{S}$, then T is a dependent set of $M_1(T)$, and so T is not a base of $\mathfrak{S}_T$. ■

$\mathfrak{S}_V$ has the same circuits as the matroid $\mathscr{M}_V = M_1(V) \oplus M_1(V^{\sim})$ on W, but another base set. Similarly to the above result, we see that any symmetric matroid is the independent intersection of matroids:

PROPOSITION 6.2. *$\mathfrak{S}$ is the independent intersection of matroids $(\mathscr{M}_V : V$ a transversal of $W)$.*

Proof. Obviously, any independent set of $\mathfrak{S}$ is an independent set of any $\mathscr{M}_V$. Conversely, if X is dependent in $\mathfrak{S}$, for a transversal V of W with $X \subseteq V$, X is dependent in $\mathscr{M}_V$. ■

REFERENCES

1 A. Bouchet, Greedy algorithm and symmetric matroids, *Math. Programming* 38:147–159 (1987).

2 ——, Representability of Δ-matroids in *Combinatorics*, Colloq. Math. Soc. János Bolyai 52, Eger, Hungary, 1987, pp. 167–182.

3 A. Bouchet, A. Dress, and T. Havel, Δ-Matroids and metroids, *Adv. Math.* 91:136–142 (1992).

4 R. Chandrasekaran and S. N. Kabadi, Pseudomatroids, *Discrete Math.* 71:205–217 (1988).

5 A. Dress and T. Havel, Some combinatorial properties of discriminants in metric vector spaces, *Adv. Math.* 62:285–312 (1986).

6 A. Duchamp, Extensions ponctuelles pour les matroïdes symétriques," *C. R. Acad. Sci. Paris Sér. I Math.* 311:663–666 (1990).

7 ——, Études de Quelques Notions et Propriétés Relatives aux Matroïdes Symétriques, Thèse, Univ. du Maine, 1991.

8 ——, Delta matroids whose fundamental graphs are bipartite, *Linear Algebra Appl.* 160:99–112 (1992).

9 I. M. Gelfand and V. V. Serganova, On the general definition of a matroid and a greedoid, *Soviet Math. Dokl.* 35:6–10 (1987).

10 B. Korte, L. Lovasz, and R. Schrader, *Greedoid Theory*, Springer-Verlag.

11 Liqun Qi, Directed submodularity, ditroids and directed submodular flows, *Math. Programming* 42:579–599 (1988).
12 E. Tardos, Generalized matroids and supermodular colouring, in *Matroid Theory (Szeged, Hungary, 1982)*, Colloq. Math. Soc. János Bolyai 40, 1985, pp. 359–382.
13 W. T. Tutte, Lectures on matroids, *J. Res. Nat. Bur. Standards* 69B:1–47 (1965).
14 D. J. A. Welsh, *Matroid Theory*, Academic, London, 1976.

Received 8 February 1993; final manuscript accepted 3 January 1994